1 Absolute convergence and the Lesbesgue Integral

After learning in class that the Cauchy distribution lacks a first moment, some have asked the following:

Why must $\int |f|\ d\mu$ exist in order for $\int f\ d\mu$ to exist?

A short answer to this question is that the requirement is part of the definition of the Lesbesgue integral.

The usual development of the Lesbesgue integral begins with the integration of indicator functions, then positive linear combinations of indicators, then non-negative functions. Once the integrals of non-negative functions f are defined, we might define:

$$f^+ = \max(f, 0) \qquad f^- = -\min(f, 0)$$

This separates a function into its positive and negative parts. (The second one is sometime written $\max(-f, 0)$.) We now have two viable non-negative integrals, for some $E \in \mathbb{R}$:

$$\int_E f^+\ d\mu \qquad \int_E f^-\ d\mu \tag{1}$$

Then, we may define the Lesbesgue integral of f to be (throughout, μ is a Lesbesgue measure):

$$\int_E f\ d\mu = \int_E f^+\ d\mu - \int_E f^-\ d\mu \tag{2}$$

Rudin (1976) discusses these integrals:

> If both integrals in (1) are finite, then (2) is finite, and we say that f is *integrable* (or *summable*) on E in the Lesbesgue sense, with respect to μ; ...
>
> This terminology may be a little confusing: If (2) is $+\infty$ or $-\infty$, then the integral of f over E is defined, although f is not integrable in the above sense of the word; f is integrable on E only if its integral over E is finite. (pp. 314-315)

The integral of the $|f|$ will be infinite whenever either integral in (1) is infinite, so we can say that

$$\int_E |f|\ d\mu < \infty \tag{3}$$

is a requirement for Lesbesgue integrability: as Hogg (1983) briefly discusses, this is called *absolute convergence*. Other notions of integration exist (Henstock-Kurzweil/Denjoy/Perron, for example), but we will not refer to them in this class.

See:

Rudin, W. (1976): *Principles of mathematical analysis.* 3ed, New York: McGraw-Hill. (Ch.11)

Hogg, R. V. (1983): *Probability and statistical inference.* New York: Macmillan. (p. 59)

Stein, E. M., and Shakarchi, R. (2005): *Real analysis : measure theory, integration, and Hilbert spaces.* Princeton: Princeton University Press.

2 Existence of E$[x^j]$

With this requirement for the existence of a Lesbesgue integral, and thus existence of a moment, it becomes clear why (for $j > 0$), if $\mathbb{E}[|x^j|] = \infty$, so must $\mathbb{E}[|x^k|]\ \forall k > j$: Since the region between -1 and 1 can make at most a contribution of 1 to $\mathbb{E}[|x^j|]$, the divergence of $\mathbb{E}[|x^j|]$ must depend on $x : |x| > 1$. But $|x^k| > |x^j|\ \forall x : |x| > 1$, so (if you like, referring to CB Thm. 2.2.5) $\mathbb{E}[|x^k|]$ must be infinite as well.