

Econ-240A (1st Half - Fall 2008)  
Note on section 2008-09-19  
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In section today (especially the second section), I lost track of why a certain inequality involving the infimum held true. Here, I try to carefully complete the argument.

We had:

$$\mathbb{P}[\exp(tX) \geq \exp(tx)] \leq \frac{\mathbb{E}[\exp(tX)]}{\exp(tx)} \quad \forall t > 0 \quad (1)$$

The question is whether, when  $\mathbb{E}[\exp(tX)]$  exists (is not infinite), this implies:

$$\mathbb{P}[\exp(tX) \geq \exp(tx)] \leq \inf_{t>0} \left( \frac{\mathbb{E}[\exp(tX)]}{\exp(tx)} \right) \quad (2)$$

First, note that the expression on the left side of the inequalities *does not depend on t*, as long as  $t$  is positive. We now write the term on the left side simply as  $\mathbb{P}$ , and we want to know whether the following is true:

$$\mathbb{P} \leq \inf_{t>0} \left( \frac{\mathbb{E}[\exp(tX)]}{\exp(tx)} \right) \quad (3)$$

If the infimum (the greatest lower bound) is actually the minimum, then there exists  $t^*$  for which:

$$\inf_{t>0} \left( \frac{\mathbb{E}[\exp(tX)]}{\exp(tx)} \right) = \frac{\mathbb{E}[\exp(t^*X)]}{\exp(t^*x)}$$

and in that case, we know that

$$\mathbb{P} \leq \frac{\mathbb{E}[\exp(t^*X)]}{\exp(t^*x)}$$

since (1) is true for all values of  $t$ .

The interesting case is when the infimum is just below all the values that the right hand side can take on. In that case, consider the alternative: if the statement (3) is false, then we must have:

$$\mathbb{P} > \inf_{t>0} \left( \frac{\mathbb{E}[\exp(tX)]}{\exp(tx)} \right) \quad (4)$$

Since (4) is a strict inequality, there must exist a value of the original function from the right hand side of inequality (1) which lies in the interval between the two sides of the inequality:

$$\begin{aligned} \mathbb{P} > \inf_{t>0} \left( \frac{\mathbb{E}[\exp(tX)]}{\exp(tx)} \right) \\ \Rightarrow \exists \tilde{t} \text{ such that } \inf_{t>0} \left( \frac{\mathbb{E}[\exp(tX)]}{\exp(tx)} \right) < \frac{\mathbb{E}[\exp(\tilde{t}X)]}{\exp(\tilde{t}x)} < \mathbb{P} \end{aligned} \quad (5)$$

If (4) is true, then there must exist  $\tilde{t}$  satisfying inequality (5); otherwise, the infimum could not be the greatest lower bound. However, from (1), we also know that

$$\frac{\mathbb{E}[\exp(\tilde{t}X)]}{\exp(\tilde{t}x)} \geq \mathbb{P}$$

which contradicts (5). Thus, since (5) can neither be true nor false, statement (4) cannot be true, and thus (3) and (2) must be true. QED.