Econ-240A (1st Half - Fall 2008)
Note on section 2008-09-19

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In section today (especially the second section), I lost track of why a certain inequality involving the infimum held true. Here, I try to carefully complete the argument.

We had:

$$\mathbb{P}[\exp(tX) \geq \exp(tx)] \leq \frac{\mathbb{E}[\exp(tX)]}{\exp(tx)} \quad \forall t > 0 \tag{1}$$

The question is whether, when $\mathbb{E}[\exp(tX)]$ exists (is not infinite), this implies:

$$\mathbb{P}[\exp(tX) \geq \exp(tx)] \leq \inf_{t>0} \left(\frac{\mathbb{E}[\exp(tX)]}{\exp(tx)} \right) \tag{2}$$

First, note that the expression on the left side of the inequalities *does not depend on* t, as long as t is positive. We now write the term on the left side simply as $\mathbb{P}$, and we want to know whether the following is true:

$$\mathbb{P} \leq \inf_{t>0} \left(\frac{\mathbb{E}[\exp(tX)]}{\exp(tx)} \right) \tag{3}$$

If the infimum (the greatest lower bound) is actually the minimum, then there exists t^* for which:

$$\inf_{t>0} \left(\frac{\mathbb{E}[\exp(tX)]}{\exp(tx)} \right) = \frac{\mathbb{E}[\exp(t^*X)]}{\exp(t^*x)}$$

and in that case, we know that

$$\mathbb{P} \leq \frac{\mathbb{E}[\exp(t^*X)]}{\exp(t^*x)}$$

since (1) is true for all values of t.

The interesting case is when the infimum is just below all the values that the right hand side can take on. In that case, consider the alternative: if the statement (3) is false, then we must have:

$$\mathbb{P} > \inf_{t>0} \left(\frac{\mathbb{E}[\exp(tX)]}{\exp(tx)} \right) \tag{4}$$

Since (4) is a strict inequality, there must exist a value of the original function from the right hand side of inequality (1) which lies in the interval between the two sides of the inequality:

$$\begin{aligned} & \mathbb{P} > \inf_{t>0} \left(\tfrac{\mathbb{E}[\exp(tX)]}{\exp(tx)} \right) \\ \Rightarrow \quad & \exists \tilde{t} \text{ such that } \inf_{t>0} \left(\tfrac{\mathbb{E}[\exp(tX)]}{\exp(tx)} \right) < \tfrac{\mathbb{E}[\exp(\tilde{t}X)]}{\exp(\tilde{t}x)} < \mathbb{P} \end{aligned} \tag{5}$$

If (4) is true, then there must exist $\tilde{t}$ satisfying inequality (5); otherwise, the infimum could not be the greatest lower bound. However, from (1), we also know that

$$\frac{\mathbb{E}[\exp(\tilde{t}X)]}{\exp(\tilde{t}x)} \geq \mathbb{P}$$

which contradicts (5). Thus, since (5) can neither be true nor false, statement (4) cannot be true, and thus (3) and (2) must be true. QED.