Econ-240A (1st Half - Fall 2009)
Section 1

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Contents

1 Sets

1.1 Styles of proof of set equality

When we want to show that two sets are equal, there are two basic techniques. Depending on the circumstance, one or the other may be most straightforward; for this example, I show both.

Suppose we want to show that:

$$B = (B \cap A) \cup (B \cap A^c)$$

1.1.1 Method 1: containment both directions

In this technique, we show that if something is an element of the expression on the left, then it must also be an element of the expression on the right. Then we show the converse: if it is an element of the expression on the right, then it must also be an element of the expression on the left. We typically appeal to the rules of logic in these arguments.

Logical Equivalences

$$\text{Identity laws}\begin{cases} p \wedge \mathbb{T} \iff p \\ p \vee \mathbb{F} \iff p \end{cases}$$

$$\text{Idempotent laws}\begin{cases} p \wedge p \iff p \\ p \vee p \iff p \end{cases}$$

$$\text{Double negation}\{\neg\neg p \iff p$$

$$\text{Commutative laws}\begin{cases} p \wedge q \iff q \wedge p \\ p \vee q \iff q \vee p \end{cases}$$

$$\text{Associative laws}\begin{cases} (p \wedge q) \wedge r \iff (p \wedge r) \wedge (q \wedge r) \\ (p \vee q) \vee r \iff (p \vee r) \vee (q \vee r) \end{cases}$$

$$\text{Distributive laws}\begin{cases} p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r) \\ p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r) \end{cases}$$

$$\text{De Morgan's laws}\begin{cases} \neg(p \wedge q) \iff \neg p \vee \neg q \\ \neg(p \vee q) \iff \neg p \wedge \neg q \end{cases}$$

To show that the left expression is (weakly) contained in right expression:

$$\begin{aligned} \text{suppose: } x \in B & \\ x \in B \wedge \mathbb{T} & \quad \text{identity} \\ x \in B \wedge (x \in A \vee x \in A^c) & \quad \text{tautology} \\ (x \in B \wedge x \in A) \vee (x \in B \wedge x \in A^c) & \quad \text{distributive property of AND } (\wedge) \end{aligned}$$

Following the implications of both halves of the expression:

$$(x \in B \wedge x \in A) \iff x \in B \cap A \Rightarrow x \in (B \cap A) \cup (B \cap A^c)$$
$$(x \in B \wedge x \in A^c) \iff x \in B \cap A^c \Rightarrow x \in (B \cap A) \cup (B \cap A^c)$$

Thus, either way, $x \in B \Rightarrow x \in (B \cap A) \cup (B \cap A^c)$, so $B \subseteq (B \cap A) \cup (B \cap A^c)$. To show that the right expression is (weakly) contained in the left expression:

$$\begin{aligned} \text{suppose: } x \in (B \cap A) \cup (B \cap A^c) & \\ x \in (B \cap A) \vee x \in (B \cap A^c) & \quad \text{definition} \end{aligned}$$

Following the implications of both halves of the expression:

$$x \in (B \cap A) \iff x \in B \wedge x \in A \Rightarrow x \in B$$
$$x \in (B \cap A^c) \iff x \in B \wedge x \in A^c \Rightarrow x \in B$$

Thus, either way, $x \in (B \cap A) \cup (B \cap A^c) \Rightarrow x \in B$, so $(B \cap A) \cup (B \cap A^c) \subseteq B$. So we have $(B \cap A) \cup (B \cap A^c) \subseteq B$ and $B \subseteq (B \cap A) \cup (B \cap A^c)$, so $B = (B \cap A) \cup (B \cap A^c)$. ■

1.1.2 Method 2: set identities

In this technique, we simply re-write the first set until it becomes the second, appealing to laws and identities of sets (corresponding closely to those of logic).

Set Identities

$$\text{Identity laws}\begin{cases} A \cap \Omega = A \\ A \cup \varnothing = A \end{cases}$$

$$\text{Idempotent laws}\begin{cases} A \cap A = A \\ A \cup A = A \end{cases}$$

$$\text{Double complementation}\{(A^c)^c = A$$

$$\text{Commutative laws}\begin{cases} A \cap B = B \cap A \\ A \cup B = B \cup A \end{cases}$$

$$\text{Associative laws}\begin{cases} A \cap (B \cap C) = (A \cap B) \cap C \\ A \cup (B \cup C) = (A \cup B) \cup C \end{cases}$$

$$\text{Distributive laws}\begin{cases} A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \end{cases}$$

$$\text{De Morgan's laws}\begin{cases} (A \cap B)^c = A^c \cup B^c \\ (A \cup B)^c = A^c \cap B^c \end{cases}$$

Thus, we can argue:

$$\begin{aligned} B &= B \cap \Omega && \text{identity} \\ &= B \cap (A \cup A^c) && \text{definition} \\ &= (B \cap A) \cup (B \cap A^c) && \text{distributive law} \end{aligned}$$

■

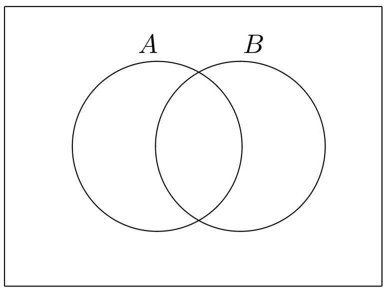
This is shorter, and is generally the method of argument I favor.

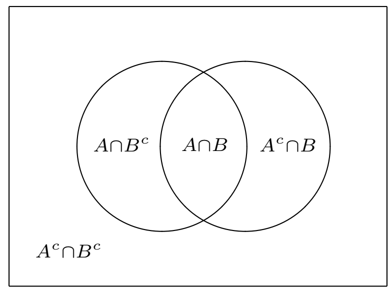
When the set descriptions in question are complicated, you may find it helpful to use Venn[1] diagrams as a guide, as in the following example.

[1] The diagrams are named after John Venn, but his work in the 1880s is preceded by a long history of logic diagrams, from Aristotle through Leibniz and Euler. Of these developments, Margaret Baron (1969) wrote: *It was, however, John Venn (1834-1923) who gave the most detailed consideration to the whole question of diagrams and it is to his credit that he took all possible steps to survey the contributions of his contemporaries, although there are few for whom he has a good word to say. The clarification brought about by Venn was, of course, associated with developments in symbolic logic and based upon the algebra of classes developed some years earlier by George Boole.* The rectangle outside the overlapping circles isn't due to Venn, however; it is due to Lewis Carroll. Baron concludes: *In some of Venn's solutions the absence of a universal class and an effective means of representing it became something of an embarassment. ... This situation appalled C. L. Dodgson who expressed himself astonished that Venn should allow the fourth subset "the rest of the Infinite plane to range about in." He, himself, provides a closed compartment for the universe of discourse so that "the Class which, under Mr. Venn's liberal sway, has been ranging about at will through Infinite Space, is suddenly dismayed to find itself 'cabin'd, cribb'd, confined' in a limited Cell like any other Class."*

Exercise 1 *Show that* $A \cup (B^c \cap A^c) = (B \cap A) \cup B^c$.

Solution 1 *First (for illustrative purposes), we draw sets* A *and* B*, overlapping in the first figure below. The four possible disjoint intersections of the sets and their complements are enumerated in the second figure below.*





Considering the original question, we see that on the left side of the equation, A *includes two of the disjoint pieces, while* $(B^c \cap A^c)$ *is only one (and it does not overlap* A*). On the other side of the proposed equality, we see that* $(B \cap A)$ *is just one disjoint piece, while* B^c *is two (and again, without overlap). The same three pieces are involved on both sides, so the equality is true, and to show it, we will need one of the expressions representing two disjoint pieces (for example,* A*) to be split into its two constituent pieces, and recombined with the other piece on the same side of the equality.*

Thus, we can argue:

$$
\begin{aligned}
A \cup (B^c \cap A^c) &= (A \cap \Omega) \cup (B^c \cap A^c) && \textit{identity} \\
&= (A \cap (B \cup B^c)) \cup (B^c \cap A^c) && \textit{definition} \\
&= ((A \cap B) \cup (A \cap B^c)) \cup (B^c \cap A^c) && \textit{distributive law} \\
&= (A \cap B) \cup ((A \cap B^c) \cup (B^c \cap A^c)) && \textit{associative law} \\
&= (B \cap A) \cup ((B^c \cap A) \cup (B^c \cap A^c)) && \textit{commutative law (in two places)} \\
&= (B \cap A) \cup (B^c \cap (A \cup A^c)) && \textit{distributive law} \\
&= (B \cap A) \cup (B^c \cap \Omega) && \textit{definition} \\
&= (B \cap A) \cup B^c && \textit{identity}
\end{aligned}
$$

■

(Venn diagrams are easy to draw for two or three sets, but beyond that, while still possible to draw, they no longer consist only of circles, and provide me no more intuitive value than would a truth table.)

References:

ROSEN, K. H. (1995): *Discrete mathematics and its applications.* 3ed, McGraw-Hill, Inc., New York, New York. (See sections 1.1 through 1.5.)

BARON, M. E. (1969): "A Note on the Historical Development of Logic Diagrams: Leibniz, Euler, Venn," *Mathematical Gazette*, 53 (383), 113-125.

2 σ-Algebras: reviewing the properties

From CB Definition 1.2.1:

a. $\mathcal{B}$ contains the empty set: $\varnothing \in \mathcal{B}$

b. $\mathcal{B}$ is closed under complementation: $A \in B \implies A^c \in B$

c. $\mathcal{B}$ is closed under countable union: $A_1, A_2, ... \in B \implies \cup_{i=1}^{\infty} A_i \in B$

Recall that in lecture, we saw the (default) discrete σ-algebra in the coin tossing example. The default sigma algebra isn't the only possible sigma algebra for the discrete case, however. Suppose we begin with a sample space $\mathcal{S} = \{A, B, C\}$, and we begin to construct a sigma-algebra $\mathcal{B}$ by asserting that it contains the subset $\{A\}$. What else must it contain?

- $\varnothing \in \mathcal{B}$, because of 1.2.1a above;
- $\{A, B, C\} \in \mathcal{B}$, because of 1.2.1b above, and because $\varnothing \in \mathcal{B}$;
- $\{B, C\} \in \mathcal{B}$, because of 1.2.1b above, and because $\{A\} \in \mathcal{B}$;

That's all we need for the sigma algebra: $\mathcal{B} = \{\varnothing, \{A\}, \{B, C\}, \{A, B, C\}\}$. Though it is not the standard choice, $\mathcal{B} = 2^{\mathcal{S}}$, it looks like the power set would if we had treated B and C to be a single element of the original sample space.

3 Independence and Combinatorics

Though this class does not focus on combinatorics, it may be a helpful tool, for example if you find yourself working on extreme value distributions and order statistics at some point in the future. Here, I begin with a quick review of events in a sample space, and I generalize to counting for cases too large to enumerate easily.

3.1 Example: 4 callers, 2 days

Consider the following setup: each of four callers will call you once on the phone over the weekend; let's call them Alan, Bronwyn, Clair, and Denis. Each call occurs independently of every other, and each is equally likely to fall on Saturday (S) or Sunday (N). First, let's write down the sample space. Each row represents an element of the sample space, which indicates each caller's choice of day:

A	**B**	**C**	**D**
S	S	S	S
S	S	S	N
S	S	N	S
S	S	N	N
S	N	S	S
S	N	S	N
S	N	N	S
S	N	N	N
N	S	S	S
N	S	S	N
N	S	N	S
N	S	N	N
N	N	S	S
N	N	S	N
N	N	N	S
N	N	N	N

The first element of this sample space is the possibility that all four callers (A, B, C, and D) call on Saturday; the second element is the possibility that all call on Saturday except Denis, who calls on Sunday; and so on. Each caller has two choices, and there are four days, so there are $2^4 = 16$ possibilities enumerated here, each occuring with probability $^1/_{16}$.

Exercise 2 *What is the probability of an even number of calls on Saturday?*

Solution 2 *We can go through the list and simply count the rows satisfying this condition:*

A	**B**	**C**	**D**	**Even-Saturday**
S	S	S	S	**Yes**
S	S	S	N	
S	S	N	S	
S	S	N	N	**Yes**
S	N	S	S	
S	N	S	N	**Yes**
S	N	N	S	**Yes**
S	N	N	N	
N	S	S	S	
N	S	S	N	**Yes**
N	S	N	S	**Yes**
N	S	N	N	
N	N	S	S	**Yes**
N	N	S	N	
N	N	N	S	
N	N	N	N	**Yes**

Eight rows satisfy this condition, so the probability of an even number of calls on Saturday is $^8/_{16} = {}^1/_2$.

We could also compute this with combinatorics: there must be either 0, 2, or 4 calls on Saturday for there to be an even number of calls on Saturday. We could use the formula for choosing k *items from a set of* n *without replacement, read as "*n *choose* k*":*

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Then we could apply it to the three cases. There is only one possible arrangement with 0 calls on Saturday; likewise for 4 calls on Saturday, and there are 6 possibilities for exactly 2 calls on Saturday:

$$\binom{4}{0} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 1; \qquad \binom{4}{4} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 1; \qquad \binom{4}{2} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1} = 6$$

Adding up: $1 + 6 + 1 = 8$, *so we also get* $^8/_{16}$

Lastly, there is a straightforward induction argument for why, with an arbitrary number of callers independently calling on one of two days, the probability of an even number of calls on a given day is $^1/_2$.

Exercise 3 *What is the probability of an even number of calls on Sunday?*

Solution 3 *Since there are an even number of calls in total, and only two days to divide them across, this is actually the same event as the previous one, so it also has probability* $^8/_{16} = {}^1/_2$.

Exercise 4 *What is the probability of Alan calling on Saturday?*

Solution 4 *This is simply* $^1/_2$*, which we know from the statement of the problem; we could also check to see that it is true in the first eight of the 16 elements of the sample space.*

Exercise 5 *Are the events in Exercises 3 and 4 independent?*

Solution 5 *For this to be true, we need:*

$$P(\text{Even \# Sat. calls AND Alan calls Sat.}) = P(\text{Even \# Sat. calls}) \cdot P(\text{Alan calls Sat.}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Checking this any number of ways, we see that the answer is **yes***.*

Exercise 6 *Are the events in Exercises 2 and 4 independent?*

Solution 6 *Since Exercises 2 and 3 are about the same events, referring to the previous answer, the answer must again be* **yes***.*

Exercise 7 *Are the events in Exercises 2 and 3 independent?*

Solution 7 *Since Exercises 2 and 3 are about the same events, the answer is* **no***. The independence of events does not obey transitivity. See CB Definition 1.3.6 through Definition 1.3.12 for further examples and details about statistical independence.*

3.2 Example: 7 callers, 7 days

This is the same setup as before, but now we also have Enrico, Fred, and George calling, and we have all seven days of the week. Again, each caller calls independently of all others, and chooses any day of the week with equal probability.

Exercise 8 *What is the probability of at least one call taking place each day?*

Solution 8 *Here, the sample space is too big to write down (with 7^7 elements). Thinking through the problem in terms of permutations and combinations:* at least' *one call each day is the same, in this case, as* exactly *one call each day, since there are equal numbers of callers and days. There are (7 choose 1) callers who could call on Monday; conditional on picking one, there are (6 choose 1) callers who could call on Tuesday; and so on. Thus there are $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ ways of this occuring. More simply, with seven callers, there are $7!$ permutations of their names where one name is assigned to each of the seven days. Thus, the probability of this event is:*

$$\frac{7!}{7^7} = \frac{5\,040}{823\,543} \approx 0.00612$$

3.3 Example: 8 callers, 7 days

This is the same setup as above, but now Hal will also call.

Exercise 9 *What is the probability of at least one call taking place each day?*

Solution 9 *Now the sample space has 7^8 elements, so we have one excess caller. Now, we need to distinguish days with two callers and days with only one. First, we pick the day with the extra caller; there are (7 choose 1) ways of doing this. Next, we pick the two callers to call on that day: there are (8 choose 2)* *ways of doing this. Finally, we assign one of the remaining six callers to each of the remaining six days; there are* 6! *ways of doing this. Thus, the probability is:*

$$\frac{7 \cdot \binom{8}{2} \cdot 6!}{7^8} = \frac{7 \cdot 28 \cdot 720}{5\,764\,801} = \frac{141\,120}{5\,764\,801} \approx 0.02448$$

Intuitively, it should make sense that the probability is going up, when compared to that in Exercise 8. This would get trickier with nine callers, because we could either have three calls on one day, or two calls on two days. We would have to add up these cases to get the probability. See Problem Set 1 for an extension to these questions, and note that you will "check" your answer via simulation.

4 CDF, PDF, and PMF: review and examples

From CB Theorem 1.5.3, a function $F(\cdot)$ is a CDF if and only if:

a. $lim_{x \to -\infty} F(x) = 0$ and $lim_{x \to \infty} F(x) = 1$.

b. $F(x)$ is a nondecreasing function of x.

c. $F(x)$ is right-continuous; that is, for every number x_0, $lim_{x \downarrow x_0} F(x) = F(x_0)$.

We will consider three putative cumulative distribution functions:

$$F_x(x) = \begin{cases} 1 - 0.5^n & \text{if } (n-1) \leq x < n \quad \text{for} \quad n \in \{1, 2, 3, ...\} \\ 0 & \text{otherwise} \end{cases}$$

$$F_y(y) = \begin{cases} 1 - \frac{1}{y^2} & \text{if } y \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F_z(z) = \begin{cases} 1 - \frac{1}{2z^2} & \text{if } z \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Exercise 10 *Confirm that each of these three functions satisfies Theorem 1.5.3, and is thus a valid CDF.*

Exercise 11 *For each CDF, does it have a PMF, PDF, or neither?*

Note that when a variable has neither PDF nor PMF, the probability measure is still Lesbesgue-integrable; the problem is just that no function can serve as a PMF (since there is density), nor can any real-valued function (from $\mathbb{R} \to \mathbb{R}^{++}$, in this case) serve as a PDF. The "Dirac Delta" measure/function, for example, which is used in some disciplines to make a "PDF" possible for mixed distributions, is not a real-valued function.

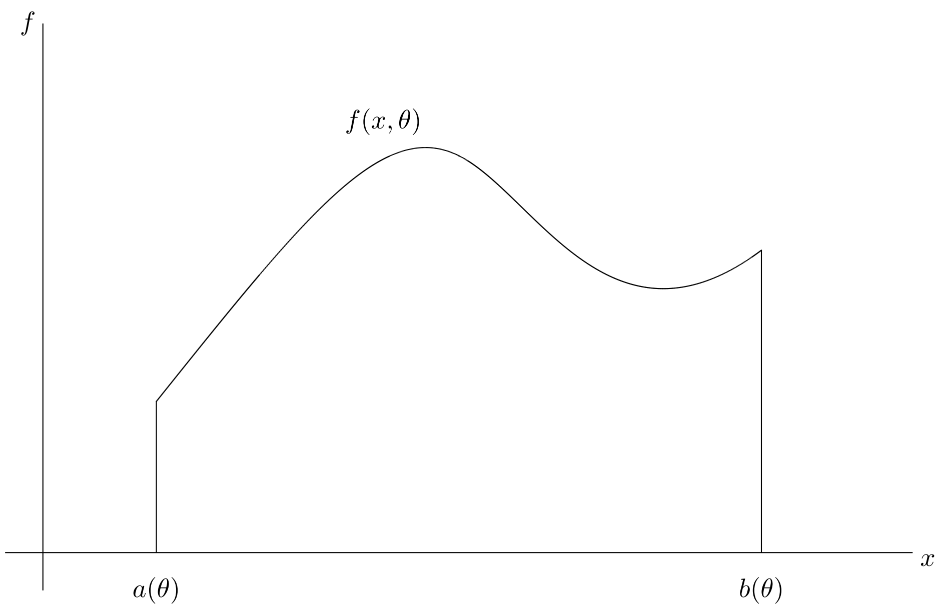
5 Leibniz' Rule

Given as Theorem 2.4.1 in Casella and Berger,[2] this rule is handy when differentiating with respect to one variable an integral with respect to another. CB provides conditions under which the rule works, and I suggest you read through the pages that follow the statement of the theorem so that you understand the cases to worry about. For example, when the integral in question explodes (does not have a finite value), it is possible that Leibniz' Rule would still give you a finite answer for the derivative of that integral with respect to another variable, even if that answer would then be meaningless. Check the conditions.

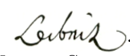
What I provide here is a graphical intuition for you to understand the rule in the cases when it works, analogous to the intuition I sketched for the product rule of derivatives in the previous section. It is not intended as a rigorous proof, rather as a mnemonic device. This will probably come in handy in cases when you want to take a derivative of an expectation with respect to a parameter of the distribution, since an expectation is an integral with respect to values of the random variable, but the parameter may enter both the limits of integration and the integrand. As such, Leibniz' Rule can turn out to be useful when working through any sort of economic model, whether macro (I encountered it back when I took 202B) or micro.[3]

Consider the graph below. The enclosed area represents the integral:

$$\int_{a(\theta)}^{b(\theta)} f(x,\theta)dx$$



For some value of θ, we can actually sketch the curve $f(x,\theta)$, and the integral is the area under it, from

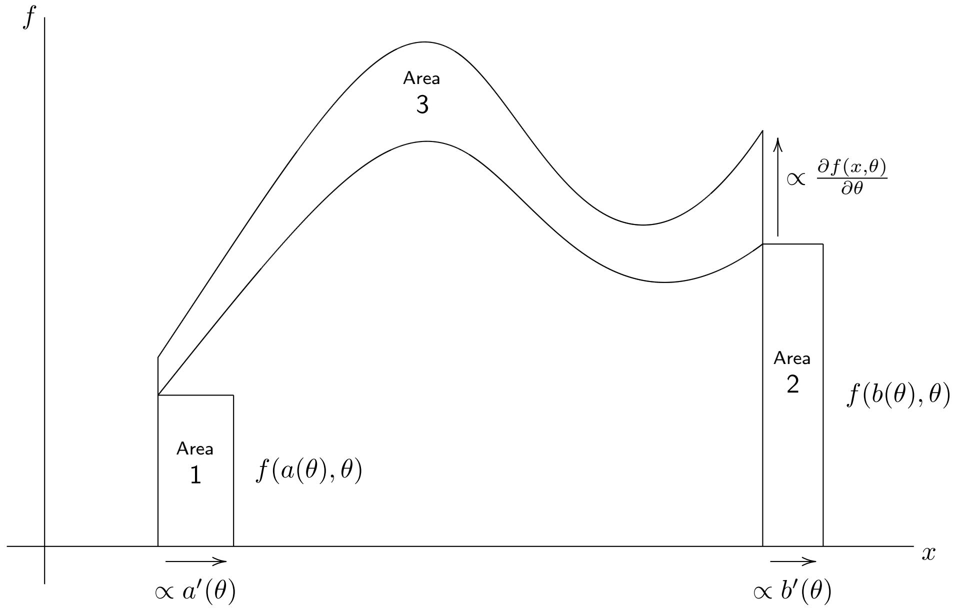
[2]CB spells it "Leibnitz." I believe the 't' is only an addition for English phonetics; he wrote his name this way: .

[3]For an example of this tool used in an economic model, take a look at equations (2) through (6) in DellaVigna, S. and Malmendier, U. (2004): "Contract Design and Self-Control: Theory and Evidence," *The Quarterly Journal of Economics*, 119 (2), 353-402.

limits $a(\theta)$ to $b(\theta)$. If we wish to understand the derivative of this integral with respect to θ,

$$\frac{d}{d\theta}\int_{a(\theta)}^{b(\theta)} f(x,\theta)dx$$

then we need to understand what will happen to the area under the curve when θ changes slightly. This change is shown in the next figure below:



First consider the rectangle labeled Area 1. A small increase in θ causes a small change in $a(\theta)$, proportional to $\frac{da(\theta)}{d\theta} = a'(\theta)$. This causes the area under the curve to *decrease* by the area of the rectangle, whose base is proportional to $a'(\theta)$, and whose height is $f(a(\theta),\theta)$; the contribution of this piece is thus proportional to $-f(a(\theta),\theta)a'(\theta)$.

Likewise, consider the rectangle labeled Area 2. A small increase in θ causes a small change in $b(\theta)$, proportional to $\frac{db(\theta)}{d\theta} = b'(\theta)$. But because this is the upper limit of integration, this causes the area under the curve to *increase* by the area of the rectangle, proportional to $+f(b(\theta),\theta)b'(\theta)$.

Finally, consider the curved area labeled Area 3. A small increase in θ causes a small change in $f(x,\theta)$ all along the curve, though the size of the change varies with x. In particular, at any point x, the change is proportional to $\frac{\partial f(x,\theta)}{\partial\theta}$. Because the change is not constant over the length, we integrate to find the area of the segment, which is thus proportional to $\int_a^b \frac{\partial f(x,\theta)}{\partial\theta}dx$.

There are small changes in area at the corners, where these areas meet one another, but these vanish in the limit as the change in θ approaches zero. Combining the three main pieces, then, (and omitting the epsilon-delta proof, as well as the details of when the limit of the integral is the integral of the limit), we have Leibniz' Rule:

$$\frac{d}{d\theta}\int_{a(\theta)}^{b(\theta)} f(x,\theta)dx = f(b(\theta),\theta)\frac{db(\theta)}{d\theta} - f(a(\theta),\theta)\frac{da(\theta)}{d\theta} + \int_{a(\theta)}^{b(\theta)} \frac{\partial f(x,\theta)}{\partial\theta}dx$$

Once again, check the text that follows CB Theorem 2.4.1 for the conditions under which this works.