Econ-240A (1st Half - Fall 2009)
Section 0

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Contents

1 Example Application: Cognitive Dissonance

We have had only one lecture so far, so there is limited material for me to build upon. Nevertheless, a recent paper's key insight relies only on the fundamentals of probability, so it serves as the main example for today. The paper is M. Keith Chen (2008): "Rationalization and cognitive dissonance: do choices affect or reflect preferences?"

From Chen's abstract: "Cognitive dissonance is one of the most influential theories in social psychology, and its oldest experiential realization is choice-induced dissonance. Since 1956, dissonance theorists have claimed that people rationalize past choices by devaluing rejected alternatives and upgrading chosen ones, an effect known as the spreading of preferences."

1.1 Testing for cognitive dissonance

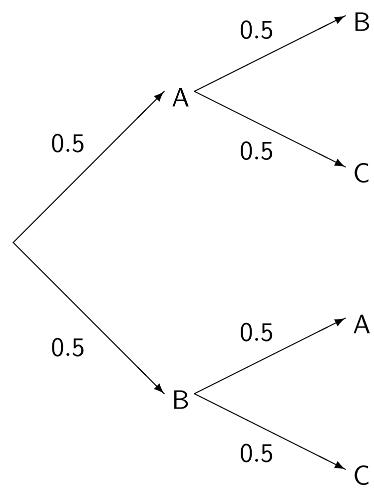
There are a variety of settings in which this phenomenon is thought to take place ("brainwashing" of various sorts are often popular examples), but one experimental setup used to test the phenomenon is quite simple, called the "free-choice paradigm," or "FCP." Here is how the experimental setup works, in two stages:

Stage 1: From Chen (2008): "In a recent FCP of this type (Egan, Santos & Bloom 2007), the experiment begins with subjects rating a number of objects on a five-point scale. Then, three objects that are rated equally (say rated 4) are chosen for use in a second stage of the experiment."

Stage 2: From Chen (2008): "In a second stage then, a subject is asked to choose between a randomly chosen two of these items, say A and B. Calling the object which the subject chooses A, the subject is then asked to choose between B (the initially rejected item), and C (the third item that was rated 4)."

Before discussing the test itself, let us consider a null hypothesis describing behavior in the absence of cognitive dissonance. The purpose of carrying out stage 1 was to identify a subset of items over which a subject is indifferent. The rating scale (1, 2, 3, 4, or 5) is meant to capture all information about the subject's preferences over the items, so that all items with a rating of 4 are equivalent in value to a participant in the experiment. Whether this is perfectly sensible or not, consider it one model of preferences. The model might go on to predict that when a participant chooses between options over which she is indifferent, the participant will choose any option with equal probability to any other, and further, than any decision is completely independent of any subsequent or prior decisions. Randomness, in this model, is introduced with every decision that is made.

Under these assumptions, behavior in the experiment should look like this:



Because each decision is independent of every other, the conditional probabilities at each stage are always 0.5. We can write out an sample space for this series of decisions:

$$S_1 = \{AB, AC, BA, BC\}$$

Formally, this is a discrete sample space, so we can use the standard σ-algebra, 2^S, with 16 elements ($\emptyset$, $\{AB\}$, etc.), and we can define the probability function on this σ-algebra by defining it over the elements in the sample space:

$$P_1(s) = \frac{1}{4} \quad \forall s \in S_1$$

Summing probabilities of disjoint events, we find:

$$P(\text{C as second choice}) = P(AC) + P(BC) = 0.5$$

In some psychology studies of cognitive dissonance, this is the null hypothesis. Then, empirically, if the fraction of study participants choosing C in the second stage is higher than (and statistically different from) 50%, the authors interpret it as evidence of cognitive dissonance: while the participants were initially indifferent between all choices, the act of initially rejecting one of two options makes one more inclined to reject it again in the future, opting for the otherwise equivalent option C more than half the time.

1.2 An economist's model

The model of behavior articulated above isn't the only possible one, however. An economist at the Yale School of Management named Keith Chen learned about this experimental protocol from colleagues of his at the primate laboratory where he studies economic behavior in monkeys, and wrote down a different model, one which views the world through the lens of preferences.

Suppose that each experimental subject has a fine-grained preference ordering over all the initial objects, and could thus rank all items perfectly. The five-level rating system used is an imperfect representation of those preferences, because even among all objects rated "4," the experimental subject might internally rate them 4.1, 4.2, and 4.3. Faced with a choice among several of these objects, the economic agent in this model will always choose the one she prefers the most, and presented with the decision again, will always make the same choice. There is no randomness in her actions, but there is randomness in "Stage 2" of the experiment, when labels are assigned to goods. This sort of randomness gives us a different view of the problem, so in order to write down a sample space, we have to consider how the random assignment of labels may relate to preferences, and that will be sufficient to determine behavior in the decisions that follow. Consider this sample space:

$$S_2 = \left\{ \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \begin{pmatrix} A \\ C \\ B \end{pmatrix}, \begin{pmatrix} B \\ A \\ C \end{pmatrix}, \begin{pmatrix} B \\ C \\ A \end{pmatrix}, \begin{pmatrix} C \\ A \\ B \end{pmatrix}, \begin{pmatrix} C \\ B \\ A \end{pmatrix} \right\}$$

There are six possible orderings of the three labels, so those orderings form the sample space: the first element represents the possibility that of the three selected items initially rated "4," the participant likes the one randomly labeled "A" the most, "B" second best, and "C" the least; and so on. For compactness, I will also refer to these six elements as s_1 through s_6. As before, since this is another discrete sample space, the standard σ-algebra will work, this time with $2^6 = 64$ elements. Since each arrangement is equally likely, we again define the probability function by specifying it over the elements of the sample space:

$$P_2(s) = \frac{1}{6} \quad \forall s \in S_2$$

Again, we can sum probabilities of disjoint events, but now we have to count more carefully. Faced with the first choice between A and B, the subject chooses A if the label arrangement is s_1, s_2, or s_5; then the subject chooses C over B if the arrangement is s_2 or s_5, but chooses B over C if it is s_1. Likewise, the subject chooses B over A in arrangements s_3, s_4, and s_6, going on to choose C over A in arrangements s_4 and s_6, but A over C in s_3. Thus:

$$P(\text{C as second choice}) = P(s_2) + P(s_5) + P(s_4) + P(s_6) = \frac{4}{6} = \frac{2}{3}$$

In this framework, *above* 2/3 of participants would have to choose item C in the second choice in order for there to be evidence of cognitive dissonance. Having written down two models of behavior, and having come to two different conclusions about what will happen in the absence of cognitive dissonance, where do we go from here?

1.3 Concluding remarks

"All models are wrong," an economist at Berkeley once said, "so the question is whether a model is wrong in the right way, or wrong in the wrong way." Both models presented here are oversimplifications of human behavior, but they may have predictive power. Are the goals of the models very different? "The key difference between psychologists and economists," suggests economist David K. Levine, "is that psychologists are interested in individual behavior while economists are interested in explaining the results of groups of people interacting." Yet in this context, both models seem to be after the same thing.

Empirically, Egan, Santos, and Bloom (2007) found that 63% of a sample of children and 60% of a sample of capuchin monkeys choose item C in the second choice of the experiment. If I have read their statistics correctly, in both cases, one can reject that the population fraction is 50%, but cannot reject that it is 2/3, and the population fraction certainly does not seem likely to be higher than 2/3. Egan, Santos, and Bloom see this as evidence of cognitive dissonance among their experimental subjects; Chen does not. To borrow a phrase from his title, do you think this is evidence that choices *affect* preferences, or only evidence that choices *reflect* preferences?

The seemingly counterintuitive conclusion of the second model—that although the labels A, B, and C were initially randomly assigned, C is more likely to be preferred in the second choice of the experiment—is explained by "selection," in economics terms: the first decision allows participants to select into different decisions in the second stage based on their preferences. This selection is responsible for the change in probability away from 1/2, and is the sort of thing that empiricists worry about all the time.

More specifically, though, this arrangement of choices over three items has more than a hundred-year history of confusing people. It is recorded as the "Bertrand's Box" problem in the late 1800s, and appeared as the "Monty Hall" problem from the 1970s through the 1990s. The free-choice paradigm described above is not the only experimental framework that psychologists have used to test for cognitive dissonance; yet the other frameworks decribed by Chen also suffer from this problem of selection distorting probabilities under a null hypothesis involving preferences, albeit in a more subtle form. For more on the Chen paper, the history of this puzzle in probability, and coverage in the popular press, see the bottom of the section website: http://economics.ozier.com/econ240a/

And remember that a simple insight can go a long way.

References:

Chen, M. K. (2008): "Rationalization and cognitive dissonance: do choices affect or reflect preferences?" *Cowles Foundation Discussion Paper* No. 1669, Yale University, New Haven, Connecticut.

Egan, L. C., Santos, L. R., and Bloom, P. (2007): "The Origins of Cognitive Dissonance: Evidence from Children and Monkeys," *Psychological Science*, 18 (11), 978-983.

2 Calculus Review

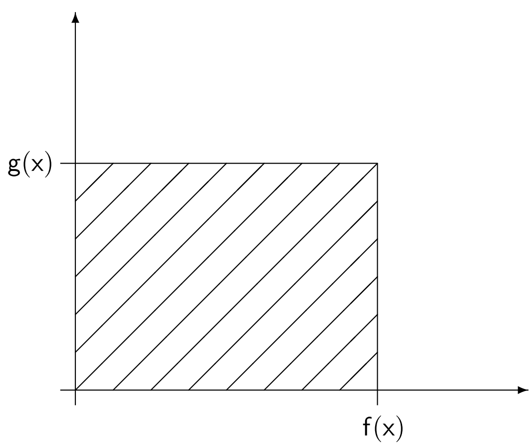
Though we have had little reason to take derivatives or integrals yet, this course will turn out to be full of them. A quick review of two integration techniques that we are likely to use repeatedly may be of help. Here, I present an explanation that very closely paraphrases parts of Stewart (1999).

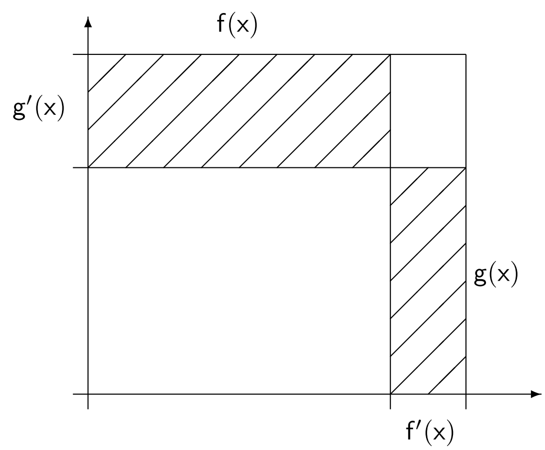
2.1 Integration by parts

We begin with a quick review of the product rule:

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + f'(x)g(x) \tag{1}$$

The graphical intuition behind the product rule: we are interested in the rate of change of $f(x)g(x)$, so picture that quantity as the area in the picture below left:





Then, if x changes slightly, $f(x)$ and $g(x)$ will each change slightly, by amounts proportional to $f'(x)$ and $g'(x)$. The new area of is the larger rectangle (pictured above right), so the change in area, proportional to the derivative of $f(x)g(x)$ is given by the three smaller rectangles. The one in the corner position shrinks faster than the other two as the change in x goes to zero (think ε^2), so consider the sum of only the two shaded ones. The change in area is proportional to $f(x)g'(x) + f'(x)g(x)$, and we have the product rule.

Satisfied with the product rule, when we take the indefinite integral of both sides of equation 1 (and thus ignore constants), the fundamental theorem of calculus makes the left side very simple:

$$f(x)g(x) = \int f(x)g'(x)dx + \int f'(x)g(x)dx \tag{2}$$

Rearranging, we get the **integration by parts** formula in equation 3, often written as in equation 4:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \tag{3}$$

$$\int u\ dv = uv - \int v\ du \tag{4}$$

This is a useful rule to have on hand when the integrand involves a product, and when application of the rule simplifies the integral so that it is more easily evaluated.

Exercise 1 *Evaluate* $\int xe^x dx$.

Solution 1 *Two ways to integrate by parts come to mind: x will either be $f(x)$ or $g'(x)$, and e^x will be the other. First, let's substitute into equation 3 with x as $g'(x)$ and with e^x as $f(x)$:*

$$\int xe^x dx = \frac{x^2}{2}e^x - \int \frac{x^2}{2}e^x dx$$

This substitution didn't help much, since the new integral is only more complicated than the one we started with. Substituting the other way, with x as $f(x)$ and with e^x as $g'(x)$, the expression does simplify:

$$\begin{aligned} \int xe^x dx &= xe^x - \int e^x dx \\ &= xe^x - e^x \end{aligned}$$

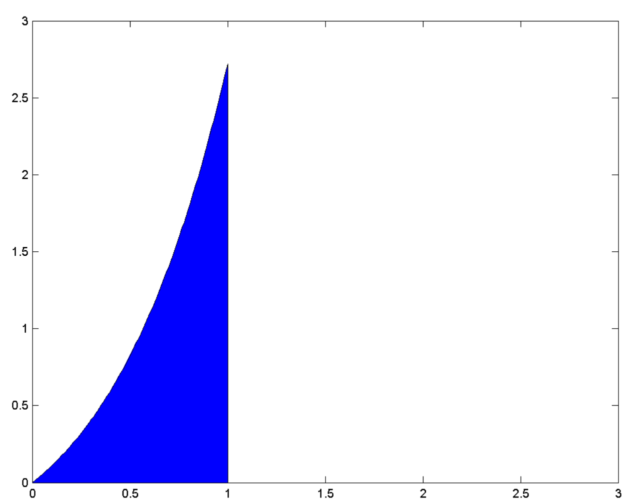
Not too bad. Were this a definite integral (with limites of integration), the technique is the same, and you can evaluate it at the end in the usual way:

Exercise 2 *Evaluate* $\int_0^1 xe^x dx$.

Solution 2 *As above, apply integration by parts, but now evaluate at the limits:*

$$\begin{aligned} \int_0^1 xe^x dx &= xe^x - e^x \Big|_0^1 \\ &= 1e^1 - e^1 - (0e^0 - e^0) \\ &= e - e - 0 - -1 \\ &= 1 \end{aligned}$$

A glance at the plot of this function confirms that the answer is reasonable:



Quite frequently, the answer does not immediately emerge after one application of integration by parts, as in the next example:

Exercise 3 *Evaluate* $\int x^2 e^x dx$.

Solution 3 *Integrating by parts, with* x^2 *as* $f(x)$ *and with* e^x *as* $g'(x)$*:*

$$\int x^2 e^x dx = x^2 e^x - \int 2xe^x dx$$

We can subsitute in our solution from Exercise 1 above:

$$\begin{aligned} &= x^2 e^x - 2\int xe^x dx \\ &= x^2 e^x - 2\left(xe^x - e^x\right) \\ &= x^2 e^x - 2xe^x + 2e^x \end{aligned}$$

2.2 Integration by substitution

This time, we begin with the chain rule:

$$\frac{d}{dx} F\left(g(x)\right) = F'\left(g(x)\right) g'(x) \tag{5}$$

Taking integrals, and applying the fundamental theorem of calculus:

$$F\left(g(x)\right) = \int F'\left(g(x)\right) g'(x) dx \tag{6}$$

This is more simply expressed in terms of the derivative of F, written in equation 7 below as f. If $u = g(x)$ is a differentiable function, and f is continuous on the range of g, then:

$$\int f\left(g(x)\right) g'(x) dx = \int f(u) du \tag{7}$$

The expression above is the formula for **integration by substitution**, where we simply substitute u for $g(x)$, and du for $g'(x)dx$. Were this a definite integral, we would have to change the limits of integration so that they are in terms of u rather than x, or evaluate in terms of x at the end. The formula for a definite integral, then, is:

$$\int_a^b f\left(g(x)\right) g'(x) dx = \int_{g(a)}^{g(b)} f(u) du \tag{8}$$

This rule is handy any time the integrant includes one function inside another; usually, the art of using the rule is in figuring out how to arrange things so that $g'(x)$ also appears in the integrand.

Exercise 4 *Evaluate* $\int \sqrt{2x+1}\, dx$.

Solution 4 *Initially, there is no* $g'(x)$ *evident in the integral, but since the inside function* $g(x)$ *is just* $2x+1$*,* $g'(x) = 2$*, so we just multiply and divide by two:*

$$\int \sqrt{2x+1}\, dx = \frac{1}{2} \int \sqrt{2x+1} \cdot 2dx$$

Now, we have $u = g(x) = 2x + 1$ *and* $du = 2dx$*, so:*

$$\begin{aligned}\int \sqrt{2x+1}\; dx &= \frac{1}{2}\int \sqrt{2x+1}\cdot 2dx \\ &= \frac{1}{2}\int \sqrt{u}\; du \\ &= \frac{1}{2}\int u^{1/2} du\end{aligned}$$

This has an easy antiderivative:

$$\frac{1}{2}\int u^{1/2} du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} = \frac{1}{3}u^{3/2} = \frac{1}{3}(2x+1)^{3/2}$$

Now, consider a definite integral:

Exercise 5 *Evaluate* $\int_0^4 \sqrt{2x+1}\; dx$.

Solution 5 *As above, we integrate by substitution. We could either integrate through and substituting* x *back in at the end, as in exercise 4, to get:*

$$\begin{aligned}\int_0^4 \sqrt{2x+1}\; dx &= \frac{1}{3}(2x+1)^{3/2}\Big|_0^4 \\ &= \frac{1}{3}(8+1)^{3/2} - \frac{1}{3}(0+1)^{3/2} \\ &= \frac{1}{3}(9)^{3/2} - \frac{1}{3}(1)^{3/2} \\ &= \frac{1}{3}(27) - \frac{1}{3}(1) \\ &= \frac{26}{3}\end{aligned}$$

Or we could have stopped short of substituting back in, and instead changed the limits of integration (as specified in equation 8 above) when substituting:

$$\begin{aligned}\int_0^4 \sqrt{2x+1}\; dx &= \frac{1}{2}\int_1^9 \sqrt{u}\; du \\ &= \frac{1}{3}u^{3/2}\Big|_1^9 \\ &= \frac{1}{3}(9)^{3/2} - \frac{1}{3}(1)^{3/2} \\ &= \frac{1}{3}(27) - \frac{1}{3}(1) \\ &= \frac{26}{3}\end{aligned}$$

We get the same answer either way, but the latter way is often more convenient; we just have to be careful to keep track of the limits of integration.

Reference:

Stewart, J. (1999): *Calculus: early transcendentals.* 4ed, Brooks/Cole Publishing Company, Pacific Grove, California. (See sections 5.5 and 7.1.)