Midterm Exam

October 17, 2007

Instructions: This is a closed book exam, but you may refer to one sheet of notes. You have 80 minutes for the exam. Answer as many questions as possible. Partial answers get partial credit. Please write legibly. *Good luck!*

Problem 1 (10 points). For *two* of the three statements below, determine whether or not the statement is correct, and give a *brief* (e.g., a bluebook page or less) justification for your answer.

(a) Let (X, Y) be a bivariate random vector with $X \sim Ber(p)$ and $Y \sim Ber(q)$ for some $0 < p, q < 1$. The random variables X and Y are independent if and only if $E(XY) = E(X) E(Y)$.

The statement is correct. The "only if" part is true for any (X, Y) (for which the expectations are well defined). Conversely, if $E(XY) = E(X) E(Y)$, then

$$P[X = 1, Y = 1] = E(XY) = E(X) E(Y) = P(X = 1) P(Y = 1),$$

where the first and last equalities use the assumption that X and Y have (marginal) Bernoulli distributions. Analogous reasoning shows that, in fact,

$$P[X = x, Y = y] = P(X = x) P(Y = y)$$

holds for any $x \in \{0, 1\}$ and any $y \in \{0, 1\}$. Therefore, the "if" part of the statement is valid, too.

(b) If X is a random variable, then

$$P(X \geq x) \leq \inf_{t>0} [\exp(-tx) M_X(t)] \qquad \forall x \in \mathbb{R},$$

where $M_X(\cdot)$ is the moment generating function of X.

The statement is correct. Indeed, $\exp(t\cdot)$ is an increasing function for any $t > 0$, so

$$P(X \geq x) = P[\exp(tX) \geq \exp(tx)] \leq \frac{E[\exp(tX)]}{\exp(tx)} = \frac{M_X(t)}{\exp(tx)}$$

for every $x \in \mathbb{R}$, where the inequality uses Chebychev's inequality.

(c) Let $\hat{\theta}$ be an estimator of $\theta \in \Theta \subseteq \mathbb{R}$ (with finite variance), let T be a sufficient statistic for θ, and define $\tilde{\theta} = E\left(\hat{\theta}|T\right)$. Then

$$E_\theta\left[\left(\tilde{\theta}-\theta\right)^2\right] \leq E_\theta\left[\left(\hat{\theta}-\theta\right)^2\right] \qquad \forall\theta \in \Theta$$

if and only if $\hat{\theta}$ is an unbiased estimator of θ.

The statement is incorrect. The "if" part is true by the Rao-Blackwell theorem. On the other hand, the inequality is valid whether or not $\hat{\theta}$ *is unbiased because*

$$\begin{aligned} E_\theta\left[\left(\tilde{\theta}-\theta\right)^2\right] &= \left[E_\theta\left(\tilde{\theta}\right)-\theta\right]^2 + Var_\theta\left(\tilde{\theta}\right) \\ &\leq \left[E_\theta\left(\hat{\theta}\right)-\theta\right]^2 + Var_\theta\left(\hat{\theta}\right) = E_\theta\left[\left(\hat{\theta}-\theta\right)^2\right], \end{aligned}$$

where the inequality uses the law of iterated expectations, which delivers the equality

$$E_\theta\left(\tilde{\theta}\right) = E_\theta\left[E\left(\hat{\theta}|T\right)\right] = E_\theta\left(\hat{\theta}\right),$$

and the conditional variance identity, which delivers the inequality

$$Var_\theta\left(\tilde{\theta}\right) = Var_\theta\left[E\left(\hat{\theta}|T\right)\right] = Var_\theta\left(\hat{\theta}\right) - E_\theta\left[Var\left(\hat{\theta}|T\right)\right] \leq Var_\theta\left(\hat{\theta}\right).$$

Problem 2 (40 points, each part receives equal weight). Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with pdf

$$f_X(x|\theta) = c(\theta)\, x \exp\left(-\frac{1}{\theta}x^2\right) 1(x > 0),$$

where $\theta \in \Theta = (0, \infty)$ is an unknown parameter, $c(\cdot)$ is a function (with argument θ), and $1(\cdot)$ is the indicator function.

(a) Show that $c(\theta) = 2/\theta$.

To solve for $c(\theta)$, *we will use the fact that if* $f_X(\cdot|\theta)$ *is a pdf, then* $\int_{-\infty}^{\infty} f_X(x|\theta)\, dx = 1$. Now,

$$\begin{aligned}
\frac{1}{c(\theta)} \int_{-\infty}^{\infty} f_X(x|\theta)\, dx &= \int_{-\infty}^{\infty} x \exp\left(-\frac{1}{\theta}x^2\right) 1(x > 0)\, dx = \int_0^{\infty} x \exp\left(-\frac{1}{\theta}x^2\right) dx \\
&= -\frac{\theta}{2} \exp\left(-\frac{1}{\theta}x^2\right)\Big|_{x=0}^{\infty} = \frac{\theta}{2},
\end{aligned}$$

so $c(\theta) = 2/\theta$, *as was to be shown.*

(b) Show that $F_X(\cdot|\theta)$, the cdf of X, is given by

$$F_X(x|\theta) = \left[1 - \exp\left(-\frac{1}{\theta}x^2\right)\right] 1(x > 0).$$

The cdf satisfies $F_X(x|\theta) = \int_{-\infty}^{x} f_X(r|\theta)\, dr$ *for any* x. *Therefore,* $F_X(x|\theta) = 0$ *when* $x \leq 0$, while

$$\begin{aligned}
F_X(x|\theta) &= \int_0^{x} f_X(r|\theta)\, dr = \int_0^{x} \frac{2}{\theta} r \exp\left(-\frac{1}{\theta}r^2\right) dr = -\exp\left(-\frac{1}{\theta}r^2\right)\Big|_{r=0}^{x} \\
&= 1 - \exp\left(-\frac{1}{\theta}x^2\right)
\end{aligned}$$

when $x > 0$.

(c) Let $Y_i = X_i^2$. Find $F_Y(\cdot|\theta)$, the cdf of Y.

Because $(\cdot)^2$ *is strictly increasing on* $(0,\infty)$, *we have:*

$$\begin{aligned} F_Y(y|\theta) &= P_\theta(Y_i \le y) = P_\theta\left[X_i^2 \le y\right] = P_\theta\left[X_i \le \sqrt{y}\right] = F_X\left[\sqrt{y}|\theta\right] \\ &= \left[1 - \exp\left(-\frac{1}{\theta}y\right)\right] 1(y>0). \end{aligned}$$

(d) Show that $E_\theta(X_i) = \sqrt{\pi\theta}/2$ and use this fact to derive a method moments estimator $\hat{\theta}_{MM}$ of θ. *[Hint: Use the fact (about the normal distribution) that*

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} x^2 \exp\left(-\frac{1}{2\sigma^2}x^2\right) dx = \sigma^2$$

for any $\sigma^2 > 0$.*]*

Using the hint (with $\sigma^2 = \theta/2$*), we have:*

$$\int_0^\infty x^2 \exp\left(-\frac{1}{\theta}x^2\right) dx = \frac{1}{2}\int_{-\infty}^{\infty} x^2 \exp\left(-\frac{1}{\theta}x^2\right) dx = \frac{\sqrt{\pi\theta}}{2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\theta}} x^2 \exp\left(-\frac{1}{\theta}x^2\right) dx = \frac{\sqrt{\pi\theta}}{2}\cdot\frac{\theta}{2},$$

where the first equation uses symmetry of $(\cdot)^2$ *and the last equality uses the hint (with* $\sigma^2 = \theta/2$*).*

As a consequence,

$$E_\theta(X_i) = \int_{-\infty}^{\infty} x f_X(x|\theta)\, dx = \frac{2}{\theta}\cdot\int_0^\infty x^2 \exp\left(-\frac{1}{\theta}x^2\right) dx = \frac{\sqrt{\pi\theta}}{2}.$$

Because $\theta = \left[2E_\theta(X_i)/\sqrt{\pi}\right]^2$, *a method moments estimator of* θ *is given by*

$$\hat{\theta}_{MM} = \frac{4}{\pi}\bar{X}^2.$$

(e) It can be shown that $E_\theta\left(X_i^2\right) = \theta$. Using this fact, find $E_\theta\left(\hat{\theta}_{MM}\right)$. Is $\hat{\theta}_{MM}$ an unbiased estimator of θ?

We have:

$$\begin{aligned} E_\theta\left(\hat{\theta}_{MM}\right) &= \frac{4}{\pi}E_\theta\left(\bar{X}^2\right) = \frac{4}{\pi}\left[E_\theta\left(\bar{X}\right)^2 + Var_\theta\left(\bar{X}\right)\right] = \frac{4}{\pi}\left[E_\theta(X_i)^2 + \frac{Var_\theta(X_i)}{n}\right] \\ &= \frac{4}{\pi}\left[E_\theta(X_i)^2 + \frac{E_\theta\left(X_i^2\right) - E_\theta(X_i)^2}{n}\right] = \frac{4}{\pi}\left[\frac{\pi\theta}{4} + \frac{1}{n}\left(\theta - \frac{\pi\theta}{4}\right)\right] \\ &= \theta\left[1 + \frac{1}{n}\left(\frac{4}{\pi} - 1\right)\right], \end{aligned}$$

implying in particular that $\hat{\theta}_{MM}$ *is a biased estimator of* θ.

(f) Let $\tilde{\theta} = nX_{(1)}^2 = n\min(X_1, \ldots, X_n)^2$. Find the cdf of $\tilde{\theta}$. Is $\tilde{\theta}$ an unbiased estimator of θ?

Let $F_{\tilde{\theta}}(\cdot|\theta)$ denote the cdf of $\tilde{\theta}$. For $r > 0$, we have (using the i.i.d. assumption):

$$\begin{aligned} F_{\tilde{\theta}}(r|\theta) &= P_\theta\left[nX_{(1)}^2 \le r\right] = 1 - P_\theta\left[nX_{(1)}^2 > r\right] = 1 - P_\theta\left[\min(X_1, \ldots, X_n) > \sqrt{r/n}\right] \\ &= 1 - P_\theta\left[X_1 > \sqrt{r/n}, \ldots, X_n > \sqrt{r/n}\right] = 1 - \prod_{i=1}^n P_\theta\left[X_i > \sqrt{r/n}\right] \\ &= 1 - \left[1 - F_X\left(\sqrt{r/n}|\theta\right)\right]^n = 1 - \left[\exp\left(-\frac{1}{\theta}\frac{r}{n}\right)\right]^n = 1 - \exp\left(-\frac{1}{\theta}r\right). \end{aligned}$$

In other words, $F_{\tilde{\theta}}(\cdot|\theta) = F_Y(\cdot|\theta)$, so

$$E_\theta\left(\tilde{\theta}\right) = E_\theta(Y_i) = E_\theta\left(X_i^2\right) = \theta,$$

implying in particular that $\tilde{\theta}$ is unbiased.

(g) Find the log likelihood function and show that the maximum likelihood estimator of θ is given by

$$\hat{\theta}_{ML} = \frac{1}{n}\sum_{i=1}^n X_i^2.$$

The log likelihood function is

$$\begin{aligned} \ell(\theta|X_1, \ldots, X_n) &= \sum_{i=1}^n \log f_X(X_i|\theta) = \sum_{i=1}^n \log\left[\frac{2}{\theta}X_i\exp\left(-\frac{1}{\theta}X_i^2\right)\underbrace{1(X_i > 0)}_{\equiv 1}\right] \\ &= -n\log\left(\frac{\theta}{2}\right) + \sum_{i=1}^n \log X_i - \frac{1}{\theta}\sum_{i=1}^n X_i^2. \end{aligned}$$

Because

$$\begin{aligned} \frac{\partial}{\partial\theta}\ell(\theta|X_1, \ldots, X_n) &= -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^n X_i^2, \\ \frac{\partial^2}{\partial\theta^2}\ell(\theta|X_1, \ldots, X_n) &= \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^n X_i^3 = -\frac{1}{\theta}\frac{\partial}{\partial\theta}\ell(\theta|X_1, \ldots, X_n) - \underbrace{\frac{1}{\theta^3}\sum_{i=1}^n X_i^3}_{>0}, \end{aligned}$$

$\hat{\theta}_{ML} = n^{-1}\sum_{i=1}^n X_i^2$ is the unique solution to the equation $\partial\ell(\theta|X_1, \ldots, X_n)/\partial\theta = 0$ and furthermore satisfies $\partial^2\ell\left(\hat{\theta}_{ML}|X_1, \ldots, X_n\right)/\partial\theta < 0$. As a consequence, $\hat{\theta}_{ML}$ is the maximum likelihood estimator of θ.

(h) Show that $\hat{\theta}_{ML}$ is a complete, sufficient statistic for θ and find a uniform minimum variance unbiased estimator of θ.

The family $\{f_X(\cdot|\theta) : \theta \in \Theta\}$ *is an exponential family. Because*

$$f_X(x|\theta) = c(\theta)\, x \exp\left(-\frac{n}{\theta}\frac{x^2}{n}\right) 1(x > 0),$$

it follows from the factorization criterion that $\hat{\theta}_{ML} = \sum_{i=1}^n \left(X_i^2/n\right)$ *is sufficient. Moreover, because*

$$\left\{-\frac{n}{\theta} : \theta \in \Theta\right\} = \left\{-\frac{n}{\theta} : 0 < \theta < \infty\right\} = (-\infty, 0)$$

contains an open set, it follows from the properties of exponential families that $\hat{\theta}_{ML}$ *is complete.*

To show that $\hat{\theta}_{ML}$ *is a uniform minimum variance unbiased estimator of* θ*, it therefore suffices to show that* $\hat{\theta}_{ML}$ *is unbiased. In turn, this result follows from the fact that* $E_\theta\left(X_i^2\right) = \theta$.