Midterm Exam

October 20, 2008

Instructions: This is a closed book exam, but you may refer to one sheet of notes. You have 80 minutes for the exam. Answer as many questions as possible. Partial answers get partial credit. Please write legibly. *Good luck!*

Problem 1 (10 points). For *two* of the three statements below, determine whether or not the statement is correct, and give a *brief* (e.g., a bluebook page or less) justification for your answer.

(a) Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with pdf $f(\cdot|\theta)$ (where $\theta \in \Theta \subseteq \mathbb{R}$ is unknown) and let $t : \mathbb{R} \to \mathbb{R}$ be a function. The statistic $\sum_{i=1}^n t(X_i)$ is sufficient (for θ) if and only if

$$f(x|\theta) = c(\theta) h(x) \exp[w(\theta) t(x)] \qquad \forall (x, \theta) \in \mathbb{R} \times \Theta$$

for some functions $c : \Theta \to \mathbb{R}_{++}$, $h : \mathbb{R} \to \mathbb{R}_+$, and $w : \Theta \to \mathbb{R}$.

(b) Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with pdf $f(\cdot|\theta)$ (where $\theta \in \Theta$ is unknown) and let $T = T(X_1, \ldots, X_n)$ be a statistic. If $\hat{\theta}$ is an unbiased estimator of θ, then $E_\theta\left(\hat{\theta}|T\right)$ is independent of (i.e., not a function of) θ if and only if T is sufficient (for θ).

(c) If (X, Y) is a bivariate random vector, then

$$P[|Y - E(Y|X)| \geq r] \leq \frac{E[Var(Y|X)]}{r^2} \leq \frac{Var(Y)}{r^2} \qquad \forall r > 0.$$

Problem 2 (40 points, each part receives equal weight). Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with pdf

$$f_X(x|\theta) = c(\theta)\, x^{-1} \exp\left[-\frac{1}{\theta}(\log x)^2\right] 1(x > 0),$$

where $\theta \in \Theta = (0, \infty)$ is an unknown parameter, $c(\cdot)$ is a function (with argument θ), and $1(\cdot)$ is the indicator function.

(a) Show that $c(\theta) = 1/\sqrt{\pi\theta}$.

(b) Let $Y_i = \log(X_i)$. Find $f_Y(\cdot|\theta)$, "the" pdf of Y. What is the distribution of Y?

(c) Show that $E_\theta(X_i) = \exp(\theta/4)$.

(d) Use the fact that $E_\theta(X_i) = \exp(\theta/4)$ to derive a method moments estimator $\hat{\theta}_{MM}$ of θ. Is $\hat{\theta}_{MM}$ an unbiased estimator of θ?

(e) Find the log likelihood function and show that the maximum likelihood estimator of θ is given by

$$\hat{\theta}_{ML} = \frac{2}{n}\sum_{i=1}^{n}(\log X_i)^2.$$

(f) Show that $\hat{\theta}_{ML}$ is a complete, sufficient statistic for θ.

(g) Find a uniform minimum variance unbiased estimator of θ and compute its variance.
[Hint: Use the fact (about the normal distribution) that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} r^4 \exp\left(-\frac{1}{2\sigma^2} r^2\right) dr = 3\sigma^4$$

for any $\sigma^2 > 0$.]

(h) Compute the Cramér-Rao bound (on the variance of unbiased estimators of θ). Is this bound attained by the estimator from (g)?