Midterm Exam

October 17, 2007

Instructions: This is a closed book exam, but you may refer to one sheet of notes. You have 80 minutes for the exam. Answer as many questions as possible. Partial answers get partial credit. Please write legibly. *Good luck!*

Problem 1 (10 points). For *two* of the three statements below, determine whether or not the statement is correct, and give a *brief* (e.g., a bluebook page or less) justification for your answer.

(a) Let (X, Y) be a bivariate random vector with $X \sim Ber\,(p)$ and $Y \sim Ber\,(q)$ for some $0 < p, q < 1$. The random variables X and Y are independent if and only if $E\,(XY) = E\,(X)\,E\,(Y)$.

(b) If X is a random variable, then

$$P\,(X \geq x) \leq \inf_{t>0}\left[\exp\left(-tx\right) M_X\left(t\right)\right] \qquad \forall x \in \mathbb{R},$$

where $M_X\,(\cdot)$ is the moment generating function of X.

(c) Let $\hat{\theta}$ be an estimator of $\theta \in \Theta \subseteq \mathbb{R}$ (with finite variance), let T be a sufficient statistic for θ, and define $\tilde{\theta} = E\left(\hat{\theta}|T\right)$. Then

$$E_\theta\left[\left(\tilde{\theta} - \theta\right)^2\right] \leq E_\theta\left[\left(\hat{\theta} - \theta\right)^2\right] \qquad \forall \theta \in \Theta$$

if and only if $\hat{\theta}$ is an unbiased estimator of θ.

Problem 2 (40 points, each part receives equal weight). Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with pdf

$$f_X(x|\theta) = c(\theta)\, x \exp\left(-\frac{1}{\theta}x^2\right) 1(x > 0),$$

where $\theta \in \Theta = (0, \infty)$ is an unknown parameter, $c(\cdot)$ is a function (with argument θ), and $1(\cdot)$ is the indicator function.

(a) Show that $c(\theta) = 2/\theta$.

(b) Show that $F_X(\cdot|\theta)$, the cdf of X, is given by

$$F_X(x|\theta) = \left[1 - \exp\left(-\frac{1}{\theta}x^2\right)\right] 1(x > 0).$$

(c) Let $Y_i = X_i^2$. Find $F_Y(\cdot|\theta)$, the cdf of Y.

(d) Show that $E_\theta(X_i) = \sqrt{\pi\theta}/2$ and use this fact to derive a method moments estimator $\hat{\theta}_{MM}$ of θ. *[Hint: Use the fact (about the normal distribution) that*

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} x^2 \exp\left(-\frac{1}{2\sigma^2}x^2\right) dx = \sigma^2$$

for any $\sigma^2 > 0$.]

(e) It can be shown that $E_\theta\left(X_i^2\right) = \theta$. Using this fact, find $E_\theta\left(\hat{\theta}_{MM}\right)$. Is $\hat{\theta}_{MM}$ an unbiased estimator of θ?

(f) Let $\tilde{\theta} = nX_{(1)}^2 = n \min(X_1, \ldots, X_n)^2$. Find the cdf of $\tilde{\theta}$. Is $\tilde{\theta}$ an unbiased estimator of θ?

(g) Find the log likelihood function and show that the maximum likelihood estimator of θ is given by

$$\hat{\theta}_{ML} = \frac{1}{n} \sum_{i=1}^{n} X_i^2.$$

(h) Show that $\hat{\theta}_{ML}$ is a complete, sufficient statistic for θ and find a uniform minimum variance unbiased estimator of θ.