Midterm Exam

October 18, 2006

Instructions: This is a closed book exam, but you may refer to one sheet of notes. You have 80 minutes for the exam. Answer as many questions as possible. Partial answers get partial credit. Please write legibly. *Good luck!*

Problem 1 (10 points). For *two* of the three statements below, determine whether or not the statement is correct, and give a *brief* (e.g., a bluebook page or less) justification for your answer.

(a) Suppose $X \sim Ber(p)$ for some $p \in (0,1)$; that is, suppose X is discrete with pmf

$$f(x|p) = p^x (1-p)^{1-x} 1(x \in \{0,1\}),$$

where $1(\cdot)$ is the indicator function. Then $E[\log f(X|q)] \leq E[\log f(X|p)]$ for every $q \in (0,1)$.

(b) Suppose $X_1, \ldots, X_n$ is a random sample with $X_i \sim Ber(p)$, where $p \in \mathcal{P} \subset (0,1)$ is unknown. Because the pmf $f(\cdot|p)$ can be written as

$$f(x|p) = 1(x \in \{0,1\})(1-p)\exp\left[\left(\frac{p}{1-p}\right)x\right],$$

the sufficient (for p) statistic $\sum_{i=1}^n X_i$ is complete if and only if $\mathcal{P}$ contains an open interval.

(c) Suppose $X_1, \ldots, X_n$ is a random sample from a continuous distribution with pdf $f(\cdot|\theta)$, where $\theta \in \Theta \subseteq \mathbb{R}$ is unknown. Then an unbiased estimator $\hat{\theta}$ of θ (with finite variance) is a uniform minimum variance unbiased estimator of θ if and only if

$$Cov_\theta\left(\tilde{\theta} - \hat{\theta}, \hat{\theta}\right) = 0 \qquad \forall \theta \in \Theta$$

for any other unbiased estimator $\tilde{\theta}$ of θ (with finite variance).

Problem 2 (40 points, each part receives equal weight). Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with cdf

$$F_X(x|\theta) = \left[a(\theta) + b(\theta)\, x^{-1/\theta}\right] 1(x > 1),$$

where $\theta \in \Theta = (0, 1)$ is an unknown parameter, $a(\cdot)$ and $b(\cdot)$ are some functions (with argument θ), and $1(\cdot)$ is the indicator function.

(a) Show that $a(\theta) = 1$ and $b(\theta) = -1$.

(b) Find $f_X(\cdot|\theta)$, "the" pdf of X. Is $\{f_x(\cdot|\theta) : \theta \in \Theta\}$ an exponential family of pdfs?

(c) Let $Y_i = \log(X_i)$. Find $F_Y(\cdot|\theta)$, the cdf of Y. Also, show that a pdf of Y is given by

$$f_Y(y|\theta) = \frac{1}{\theta} \exp\left(-\frac{1}{\theta} y\right) 1(y > 0).$$

(d) Show that $E(X_i) = 1/(1-\theta)$ and use this fact to derive a method moments estimator $\hat{\theta}_{MM,X}$ of θ. Is $\hat{\theta}_{MM,X}$ an unbiased estimator of θ?

(e) Show that $E(Y_i) = \theta$ and use this fact to derive a method moments estimator $\hat{\theta}_{MM,Y}$ of θ. Is $\hat{\theta}_{MM,Y}$ an unbiased estimator of θ?

(f) Find the log likelihood function and show that the maximum likelihood estimator of θ is given by

$$\hat{\theta}_{ML} = \frac{1}{n} \sum_{i=1}^{n} \log(X_i).$$

(g) Show that $\hat{\theta}_{ML}$ is a complete, sufficient statistic for θ and find a uniform minimum variance unbiased estimator of θ.

(h) Compute the Cramér-Rao bound (on the variance of unbiased estimators of θ). Is this bound attained by the estimator from (g)?